PMM U.S.S.R.,Vol.51,No.2,pp.265-268,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

A GEOMETRICAL MODEL OF BRITTLE FRACTURE UNDER CREEP*

A.A. VAKULENKO and V.YA. KREINOVICH

A geometric model is proposed for the microstructure of a material in a latent stage of fracture under creep. The model, based on results of an experimental study of fracture under creep, results in fractal geometry: a set equal to the union of all micropores has a fractional Hausdorff dimensionality that depends on the material and its testing conditions.

The mechanism of the latent stage of brittle fracture under creep is well-known, including as it does the formation, growth, and coalescence of microcavities that result in the evolution of macrocracks which also separate the specimens into parts /1/. Under brittle fracture conditions the typical shape of the microcavities is micropores whose dimensions in different directions are quantities of the same order /2, 3/. Such fracture processes are extremely stochastic; in particular, micropores in specimens identical from the macroscopic viewpoint being tested under identical conditions and distributed differently.

To describe the latent fracture stage under creep conditions, a scalar damage parameter /4, 5/ is used below. At the microlevel this parameter characterises the "disintegration" of the material microstructure which is the result of the formation, growth and coalescence of micropores. The relative change in the specimen density, which usually does not exceed 1-2%, is the fundamental estimate of material disintegration at the microlevel /6/. However, tests measuring the magnitude of the change in material density at a certain time of the creep process are inadequate for verifying the theory of fracture. Because of this, as a rule, the verification is performed according to data on the time to specimen macrofracture (its separation into parts) /2, 3/, which indeed underlie the construction of a geometrical model of brittle fracture. It has been shown /2, 3/ that the times to macrofracture under identical external conditions change from speciment to specimen, thereby reflecting the stochastic nature of the material microstructure.

1. According to the usual representations of the mechanics of a continuous medium /7/, if the specimen is separated into domains (or subvolumes Δ_k $(k = 1, 2, \ldots, M)$), whose dimensions are much less than the specimen dimensions and much greater than the dimensions of the material microinhomogeneity at a certain time of the process, the fracture processes in each of the subvolumes are independent of analogous processes in the other subvolumes. In conformity with the noted brittle fracture mechanism under creep, the probability of fracture of each subvolume Δ_k under given external conditions is determined uniquely by the value of the damage parameter $\omega(\Delta_k)$ $(k = 1, 2, \ldots M)$ of this subvolume.

To describe the connection between specimen fracture and fracture of the individual subvolumes, we follow /8/ and use the "weakest link" principle: the fracture of a certain domain occurs if and only if its most defective part is fractured. In the case of fracture because of micropore cumulation, this principle means that the subvolume is fractured because of its most disintegrated part. In conformity with this, the damage parameter $\omega(\Delta_k)$ of the subvolume Δ_k $(k = 1, 2, \ldots M)$ equals the greatest of the values $\omega(x)$ for all points x in the subvolume Δ_k:

$$\omega(\Delta_k) = \max_{x \in \Delta_k} \omega(x) \tag{1.1}$$

To determine $\omega(x)$ we use the physical nature of this parameter. Disintegration appears at the macrolevel in the form of a residual change in volume, whose fundamental role for cold "athermal" plasticity processes was established in /9/. Since the residual change in volume in creep processes is very much more significant, it is natural to take a function of the relative change in volume ε_v as the damage parameter. The simplest and physically most intelligent formulas to describe brittle fracture under creep can be obtained when using the expression /4/

$$\omega = \ln(1 + \varepsilon_v^p), \quad \varepsilon_v = (dv - dv_0)/dv_0 \tag{1.2}$$

where ε_v^p is the inelastic part of ε_v, dv is the body volume element at a certain time of the process, and dv_0 is the same volume element at the initial time. For small $\varepsilon_v^p ((\varepsilon_v^p)^2 \ll 1)$ $\omega \simeq \varepsilon_v^p$,

Prikl.Matem.Mekhan.,51,2,341-345,1987

but at the microlevel ε_v^p can also not be small.

By virtue of the stochastic nature of brittle fracture, the field $\omega(x)$ varies in a random manner from subvolume to subvolume. We will estimate the expectation $\langle\omega\rangle_\alpha$ of the damage for subvolumes of a certain fixed dimension α.

The quantity $\langle\omega\rangle_\alpha$ depends on the material, the external conditions (the magnitudes of the applied tensile force and temperature) and the times of action of these external conditions. If the material and external conditions are fixed, then $\langle\omega\rangle_\alpha$ is determined just by the time.

To estimate $\langle\omega\rangle_\alpha$ we divide the whole specimen into subvolumes of linear dimension α. The total number of such subvolumes M is large, the fracture processes occurring therein are independent, and with high accuracy $\langle\omega\rangle_\alpha$ agrees with the arithmetic mean of the subvolume by the law of large numbers, i.e.,

$$\langle\omega\rangle_\alpha = \frac{1}{M}\sum_{k=1}^{M}\omega(\Delta_k)$$

Since damage in the form of micropores is accumulated in an irreversible manner during brittle fracture, then for any α the $\langle\omega\rangle_\alpha$ depends monotonically on the time t. This means that $\langle\omega\rangle_\alpha$ uniquely determines the quantity t which in turn uniquely determines $\langle\omega\rangle_\alpha$ for any linear dimensions β, therefore, $\langle\omega\rangle_\beta$ is a function of $\langle\omega\rangle_\alpha$. The form of this function certainly depends on the material, the external conditions, etc.

The relative change in the specimen volume is small (as already mentioned, it reaches 1-2%), and consequently, we can confine ourselves to linear terms in the expansion of $\langle\omega\rangle_\beta$ in $\langle\omega\rangle_\alpha$. Since fracture processes are considered such that the inquality $\omega(\Delta_k) \geqslant 0$, is always satisfied, then for $\langle\omega\rangle_\alpha = 0$ for all subvolumes Δ_k $(k = 1, 2, \ldots, M)$ $\omega(\Delta_k) = 0$ and there is no damage in the specimen. In this case, for any other linear dimension β the condition $\langle\omega\rangle_\beta = 0$ is also satisfied. It hence follows that the linear terms in the expansion of $\langle\omega\rangle_\beta$ in terms of $\langle\omega\rangle_\alpha$ take the form

$$\langle\omega\rangle_\beta = c(\beta, \alpha)\langle\omega\rangle_\alpha \tag{1.3}$$

There is no "isolated" length in the theory of continuous media (with the exception of a moment theory of such a medium), consequently, the values of all the dimensionless combinations of the body characteristics, in particular, should be independent of whatever units these characteristics are measured in. The passage to a unit of length λ times small takes α over into $\lambda\alpha$ and $\langle\omega\rangle_\alpha$ into $\langle\omega\rangle_{\lambda\alpha}$ and the scale invariance noted results in the equality $c(\lambda\beta, \lambda\alpha) = c(\beta, \alpha)$ for any positive λ, β, α. If we set $\lambda = \alpha^{-1}$ in this equality, where the quantity α is measured in a certain system of unity, we obtain

$$c(\beta, \alpha) = c_1(\beta),\ c_1(\beta) \equiv c(\beta, 1) \tag{1.4}$$

Substituting (1.4) into (1.3) we have

$$\langle\omega\rangle_\beta = c_1(\beta/\alpha)\langle\omega\rangle_\alpha \tag{1.5}$$

For any positive number r with $\alpha = 1$ and $\beta = r$ we obtain from expression (1.5) that $\langle\omega\rangle_r = c_1(r)\langle\omega\rangle_1$. Setting $r = \beta$ and $r = \alpha$ here, we have relationships from which after substitution into (1.5) and later reduction by $\langle\omega\rangle_1$ we obtain $c_1(\beta) = c_1(\beta/\alpha)\,c_1(\alpha)$. Hence after substituting $\beta = \eta\xi, \alpha = \xi$ the functional equation $c_1(\eta\xi) = c_1(\eta)\,c_1(\xi)$ follows. As is known /10/, the solution of this equation has the form $c_1(\eta) = \eta^s$ and since $\eta = \beta/\alpha$ then for constant A and s

$$\langle\omega\rangle_\beta = A\beta^s, \quad c(\beta, \alpha) = (\beta/\alpha)^s \tag{1.6}$$

It has been shown /5/ that when the initial specimen is partitioned into l subvolumes, the values of $\langle\omega\rangle_\alpha$ diminish by a factor of $l^{1/n}$, where n is a constant for given stress and temperature levels.

The parameter n and s are related as follows: $s = 3/n$. In fact, if a subvolume of linear dimension β and volume approximately equal to β^3 is divided into subvolumes of any other linear dimension α (and a volume approximately equal to α^3), then, the number of subvolumes of dimension α being obtained equals approximately $(\beta/\alpha)^3$. Therefore, the equalities $\langle\omega\rangle_\beta = l^{1/n}$, $\langle\omega\rangle_\alpha = (\beta/\alpha)^{3/n}\langle\omega\rangle_\alpha$, hold, and it follows from (1.3) and (1.6) that $s = 3/n$.

2. To construct the geometrical model of the set μ which is a union of all micropores of the specimen, we introduce its volume V_μ as a natural characteristic of this set. The direct determination of V_μ is associated with attempts to measure the density, which were inadequate, as was mentioned above. We estimate V_μ by taking into account that the fundamental contribution to V_μ is from the inelastic part ε_v, i.e.,

$$V_\mu = \int_V \varepsilon_v^p(x)\,dv \tag{2.1}$$

From relationship (1.2) we have $\varepsilon_v^p = e^\omega - 1$, and for small ε_v^p the approximation $\varepsilon_v^p \simeq \omega$ is valid. The quantity ε_v^p is small on the average for the whole specimen, consequently, we replace ε_v^p by $\omega(\mathbf{x})$ in (2.1) and we write it as follows:

$$V_\mu = \int_v \omega(\mathbf{x})\, dv \tag{2.2}$$

To estimate V_μ we separate the specimen of volume v into M_0 identical parts (subvolumes) of the linear dimension α and volume v/M_0, respectively. In conformity with the approach taken in the mechanics of a continuous medium, the field $\omega(\mathbf{x})$ within each subvolume Δ_k $(k = 1, 2, \ldots, M_0)$ can be considered homogeneous. It then follows from (1.1) that $\omega(\mathbf{x}) = \omega(\Delta_k)$ for $\mathbf{x} \in \Delta_k$ and we hence have

$$V_\mu = \frac{v}{M_0} \sum_{k=1}^{M_0} \omega(\Delta_k)$$

Taking (1.7) into account, we obtain (C_μ is a constant)

$$V_\mu(\alpha) = C_\mu v \alpha^s, \quad s = 3/n \tag{2.3}$$

According to the analysis of experimental data performed in /5/, $n \geqslant 1$ and, therefore $0 \leqslant s \leqslant 3$, which implies an infinitesimal decrease in $V_\mu(\alpha)$ as $\alpha \to 0$ according to the relationship (2.3).

The concept of a Hausdorff measure is introduced for the geometrical description of a set of the type μ (see /11/, say).

We recall that the finite set $\mathbf{x}_1, \mathbf{x}_1, \ldots, \mathbf{x}_k$ is called on α-mesh of the set $\mathbf{B}$ if any point from $\mathbf{B}$ differs by not more than $\alpha > 0$ from one of the points $\mathbf{x}_i$ $(1 \leqslant i \leqslant k)$. For the very same set there exist α-meshes from a different number of points. We let $N_\alpha(\mathbf{B})$ denote the numer of points in that one of the α-meshes for $\mathbf{B}$ in which the total number of points is minimal.

It turns out that the relationship $N_\alpha(\mathbf{B}) \simeq C_3\alpha^{-3}$ is satisfied asymptotically for bodies of non-zero volume, where the quantity C_3 is proportional to the volume. For smooth surfaces $N_\alpha(\mathbf{B}) \simeq C_2\alpha^{-2}$, where the quantity C_2 is proportional to the surface area, while for segments of curves $N_\alpha(\mathbf{B}) \simeq C_1\alpha$, where the quantity C_1 is proportional to the curve length. Extending these three cases, for $N_\alpha(\mathbf{B}) \simeq C_\gamma\alpha^{-\gamma}$ it is natural to call the quantity γ the dimensionality of the set $\mathbf{B}$ (or the Hausdorff dimensionality), and the coefficient C_γ the Hausdorff measure of the set $\mathbf{B}$. These concepts can be defined even in the case of an arbitrary asymptotic $N_\alpha(\mathbf{B})$ /11/.

In α-mesh terms, the estimate of the volume is the total volume of the α-neighbourhoods of all points of this α-mesh, i.e., $C_\mu\alpha^3 N_\alpha(\mu)$, where C_μ is a constant. Equating this expression to the right side of (2.3), we obtain

$$N_\alpha(\mu) \simeq \text{const} \cdot \alpha^{s-3} \tag{2.4}$$

It follows from (2.4) that the set μ has a fractional Hausdorff dimensionality equal to $\gamma = 3 - 3/n$.

As is seen from the table, the quantity γ depends very much on the material. For the very same material, the values of γ grows for an unchanged temperature T as the stress σ decreases. The value of T noticeably influences the magnitude of this growth.

C—Cr—V [3],	$T = 748$ K	$T = 848$ K
σ, kg/mm^2	22 20 18	12 10 8
γ	0.24 0.56 1.24	1.32 1.39 2.22
C—Cr—V—Mo [3],	$T = 848$ K	Al—2S [2], $T = 753$ K
σ, kg/mm^2	20 18 14	0.41 0.36 0.30
γ	1.30 1.32 1.42	1.82 1.94 2.14

The micropore concentration /6/ is sometimes presented as a material damage characteristic in experimental works studying the microstructure of material during fracture because of creep. In this model the concentration is proportional to $N_\alpha(\mu)$, where α is the greatest micropore dimension distinguishable in a given experiment. By virtue of (2.4), the concentration equals $\text{const} \cdot \alpha^{-\gamma}$. Therefore, knowing the concentration for two different α (different enlargements), we obtain the invariant characteristics γ and const, describing fracture under creep.

3. A large number of models describing crack growth during the brittle fracture of solid deformable bodies exists /12/. In this paper the brittle fracture of metals during creep, which occurs under elevated temperature conditions ($\simeq 0.5$ the melting point) and reduced stresses with respect to the yield point at a given temperature is considered. The time to macrofracture (boy separation into parts) is here quite long (measured in years) and the main

part of the time of the fracture process is occupied by the latent fracture stage. The time this stage terminates is taken approximately as the time to specimen macrofracture. Separation of the fracture process under creep into two stages for metal bodies, the latent and the mainline crack propagation stage is customary for the theory of brittle fracture under creep /7/. In this paper the latent fracture stage is considered during which the random formation and development of micropores occurs over the specimen body.

At the macrolevel the stochastic nature of such fracture appears as the spread in the macrocharacteristics of the fracture process during creep and, in particular, as the time to macrofracture. The spread in the time to macrofracture is the more significant the lower the level of the given tensile stress for an invariant temperature /2/. A Weibull distribution of the time to macrofracture is proposed to describe this spread for sufficiently low stress levels /5, 7/. It follows from the results of creep strength tests under the above-mentioned conditions that the volume of samples of the time to macrofracture does not exceed 20 values for fixed stresses and temperature. Two parameters of the Weibull distribution of the time to macrofracture in the form $1-\exp(-A_0t^n)$ (t is the time and A_0, n are distribution parameters) are found by the maximum-likelihood method /5/.

The model developed in this paper enables us to determine the shape parameter n from the results of investigating microcavities in the plane of the specimen middle transverse section, made at a time close to the time of termination of the latent specimen creep stage /13/. The dependence of the relative number of micropores whose diameter is greater than or equal to α, on α is established by the concentration of the micropores in the plane of the section. To a first approximation this dependence is approximated by the power function $C_0\alpha^{-b}$, where C_0 and b are constants. Sing b is the dimensionality of the set which is the intersection of the set μ and a plane in three-dimensional space, then $b=\gamma+2-3$ /11/, and therefore $\gamma=b+1$. It follows from (2.4) that $n=3/(3-\gamma)$, and we hence have $n=3/(2-b)$.

The second parameter of the time-to-facture distribution A_0 is determined for known n by the maximum likelihood method from the relationship

$$A_0=K\left(\sum_{i=1}^{K}t_i^n\right)^{-1}$$

where K is the volume of samples, and t_i is the i–th value of the time to macrofracture under fixed external conditions. For known n the value of A_0 is more exact than when both parameters are unknown and found from one sampling /14/.

It follows from the above that the geometrical model of brittle fracture under creep provides a basis, to a first approximation, for utilizing the results of microstructural micropore investigations to describe the spread in the time to macrofracture of metal specimens.

REFERENCES

1. GRANT N., Fracture under high-temperature creep conditions, Fracture, 3, /Russian translation/ Mir, Moscow, 1976.
2. PHILLIPS C.W. and SINNOTT M.J., A statistical study of the stress-rupture test, Trans. Amer. Cos. Metals, 46, 1954.
3. PRNKA T. and FOLDYNA V., The creep properties of low alloy Cr-Mo-V steels with low carbon content. High Temperature Properties of Steels, Iron and Steel Inst., London, 1967.
4. VAKULENKO A.A. (Jr.), On brittle fracture under creep, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 6, 1982.
5. VAKULENKO A.A., On the statistics of brittle fracture under creep, Problemy Prochnosti, 10, 1984.
6. PIATTI G., LUBEK R. and MATERA R., Cavitation, Euro-spectra, 11, 4, 1972.
7. BOLOTIN V.V., On the theory of slow fracture, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 1, 1981.
8. KONTOROVA T.A., Statistical theory of strength, Zh. Tekh. Fiz., 10, 11, 1940.
9. NOVOZHILOV V.V., On plastic disintegration, PMM, 29, 4, 1965.
10. BELLMAN R. and COOK K.L., Differential-Difference Equations /Russian translation/, Mir, Moscow, 1967.
11. MANDELBROT B., Fractals, Form, Chance, Dimension. Freeman, New York, 1977.
12. CHEREPANOV G.P., Mechanics of Brittle Fracture, Nauka, Moscow, 1974.
13. CHEN I.W. and ARGON A.S., Creep cavitation in 304 stainless steel, Acta Metallurgica, 29, 7, 1981.
14. KLEBANOV L.B. and MELAMED I.A., Several notes on Fisher information in the presence of nuisance parameters, Math. Operationsforsch. und Statistics, 9, 1, 1978.

Translated by M.D.F.